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On the Theorem of Power Expended in Continuum Mechanics

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We show that the theorem of power expended may be claimed for unbounded bodies, under mild hypotheses on the behavior at infinity of the velocity, the density, and the stress fields. © 1985 Academic Press, Inc.

1. INTRODUCTION

Let E be an n-dimensional euclidean space, let B be a body, i.e., a regular region in E[1], and let $\{\mathbf{0}, \mathbf{e}_i\}$ be an orthonormal reference frame in E. As known, the theorem of power expended assures that, if Ω is any bounded, fixed region contained at each instant on B, then

$$\frac{1}{2}\int_\Omega \rho v^2 d\Omega = \frac{1}{2}\int_\Omega \rho_0 v_0^2 d\Omega + \int_0^t ds\left\{\int_\Omega \mathbf{b}\cdot\mathbf{v}\, d\Omega - \int_\Omega \mathbf{T}\cdot\mathbf{D}\, d\Omega + \int_{\partial\Omega}\left(\mathbf{v}\cdot\mathbf{T}\mathbf{n} - \frac{1}{2}\rho v^2 \mathbf{v}\cdot\mathbf{n}\right) d\Sigma\right\} \tag{1}$$

where the symbols have their usual meaning and the functions are supposed smooth [1].

This result poses the following question: *Does relation* (1) *hold if B and Ω are both unbounded, the body and surface forces are summable respectively on Ω and $\partial\Omega$, and the initial energy is finite*?

The following example [2] shows that such a generalization requires suitable restrictions upon the behavior at infinity of the flow $(\mathbf{v}, \rho, \mathbf{T})$. For homogeneous, incompressible, inviscid fluids ($\mathbf{T} = -p\mathbf{I}$), (1) takes the form

$$\tfrac{1}{2}\int_\Omega \rho v^2 d\Omega = \tfrac{1}{2}\int_\Omega \rho v_0^2\, d\Sigma + \int_0^t ds\left\{\int_\Omega \mathbf{b}\cdot\mathbf{v}\, d\Omega - \int_{\partial\Omega} (p + \tfrac{1}{2}\rho v^2)\, \mathbf{v}\cdot\mathbf{n}\, d\Sigma\right\}. \tag{2}$$

[1] Throughout this paper we follow the graphic conventions of [1], to which we refer for any symbols not explained here.

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A simple consequence of (2) is the uniqueness of the null solution $(\mathbf{0}, \rho, -c\mathbf{I})$, with $c = \text{const}$, under the boundary condition $\mathbf{v} \cdot \mathbf{n} \equiv 0$. On the other hand, if $B \equiv E$ and $\mathbf{b} \equiv \mathbf{0}$, the initial boundary-value problem for the system of incompressible inviscid fluids admits the solution $(\sum_i t^2 \mathbf{e}_i, \rho, -2t \sum_i x_i \mathbf{I})$ as well as the trivial one. Hence, a necessary condition in order that (1) hold is $p = |\mathbf{T}| = o(r)$, with $r = |\mathbf{x} - \mathbf{0}|$.

A second restriction involves the velocity $\mathbf{v}$. Let $(\mathbf{v}, \rho, \mathbf{T})$ be a steady flow, let S_R be the sphere of radius R centered at $\mathbf{0}$, and let $\mathbf{v}_r(R) = \sup\{|\mathbf{v} \cdot \mathbf{e}_r|, \mathbf{x} \in S_R \cap \Omega, r\mathbf{e}_r = \mathbf{x} - \mathbf{0}\}$. A physically reasonable request is that the integral $\int_0^{+\infty} dR/v_r$ is infinite.

Thus, as a first step, we may ask ourselves whether (1) holds for solutions $(\mathbf{v}, \rho, \mathbf{T})$ to the system

$$\left.\begin{aligned} \rho\dot{\mathbf{v}} &= \operatorname{div} \mathbf{T} + \mathbf{b} \\ \dot{\rho} + \operatorname{div} \mathbf{v} &= 0 \end{aligned}\right\} \quad \text{on } Q = \Omega \times (0, +\infty) \tag{3}$$

under the hypotheses

$$\begin{gathered} \text{there exists a positive and smooth function } q(r) \text{ on } [0, +\infty) \\ \text{such that } q'(r) > 0,\ \lim_{r \to +\infty} q(r) = +\infty,\ q'(r)\,|\mathbf{v} \cdot \mathbf{e}_r| = O(1) \\ \exists f(r) \in C((0, +\infty)), \qquad f(r) = O(r), \qquad \rho^{-1/2}\,|\mathbf{T}| = o(f(r)). \end{gathered} \tag{4}$$

The aim of the present work is just to give a partly affirmative answer to the above question. In fact, denoting by Ξ the whole set $(\mathbf{v}, \rho, \mathbf{T})$ of solutions to system (3) such that[2]

$$\begin{aligned} &\rho \in C(\bar{Q}) \\ &\mathbf{v} \in \mathbf{C}_1^1(Q) \cap \mathbf{C}_1(\bar{Q}) \\ &\mathbf{T} \in \mathbf{C}_2^1(Q) \cap \mathbf{C}_2(\bar{Q}),\ \mathbf{T} \cdot \mathbf{D} \geqslant 0 \end{aligned}$$

we shall prove the following

THEOREM. *Let B be an unbounded body, let $(\mathbf{v}, \rho, \mathbf{T}) \in \Xi$, and assume that*

[2] The set of functions defined on $G \subseteq E \times [0, +\infty)$ and differentiable up to the order m $(\in N_0)$ inclusive will be denoted by $C^m(G)$ $(C^0(G) \equiv C(G))$ and we agree to use the symbol $\mathbf{C}_k^m(G)$ for the set of tensor-valued functions of kth order, whose components belong to $C^m(G)$.

(4) *is satisfied with* $f^2(r) = r^{1-n}[q'(r)]^{-1}$. *If* Ω *is any fixed region contained on* B, $\forall t \geqslant 0$, *and*

$$\int_\Omega \rho_0 v_0^2 \, d\Omega < +\infty, \int_\Omega b^2/\rho \, d\Omega, \int_{\partial\Omega} |\mathbf{v}\cdot\mathbf{Tn} - \tfrac{1}{2}\rho v^2 \mathbf{v}\cdot\mathbf{n}| \, d\Sigma < +\infty, \ \forall t \geqslant 0$$

then (1) *holds.*

2. Proof of the Theorem

Consider the function

$$g(\mathbf{x}, s) = g_\delta(\psi(\mathbf{x}, t))$$

where g_δ is a C^∞ function on $\mathbb{R}$ verifying ($\delta > 0$)

$$\begin{aligned} g_\delta(\xi) &= 0 \qquad \text{on } (-\infty, -\delta] \\ g_\delta(\xi) &= 1 \qquad \text{on } [\delta, +\infty) \\ g_\delta'(\xi) &\geqslant 0 \qquad \text{on } \mathbb{R} \end{aligned}$$

and

$$\psi(\mathbf{x}, s) = c^{-1}[q(R) + c(\tau - s) - q(r)], \qquad R, \tau > 0.$$

The support of g, defined by the inequality

$$q(r) \leqslant q(R) + c(\tau - s),$$

is certainly compact on $E \times [0, \tau]$, $\forall R$. Moreover it is equal to the set

$$\chi_\delta = \{(\mathbf{x}, s) \in E \times [0, +\infty) : r \leqslant q^{-1}[q(R) + c(\tau - s + \delta)], s \leqslant \tau\}.$$

Further, $g \in C^\infty(E \times [0, \tau])$ even if ψ is not differentiable at $\mathbf{0}$. Indeed, in the set

$$\{(\mathbf{x}, s) \in E \times [0, +\infty) : r \leqslant q^{-1}[q(R) + c(\tau - s - \delta)], s \leqslant \tau\}$$

$g \equiv 1$. Hence, if we choose δ suitably small, we may conclude that g is differentiable at $\mathbf{0}$.

Multiply both sides of $(3)_1$ by $g\mathbf{v}$ and recall the following vector identities:

$$\begin{aligned} \tfrac{1}{2} g\rho\mathbf{v}\cdot\operatorname{grad} v^2 &= \operatorname{div}\{\tfrac{1}{2} g\rho v^2 \mathbf{v}\} - \tfrac{1}{2} g v^2 \operatorname{div}(\rho\mathbf{v}) - \tfrac{1}{2}\rho v^2 \mathbf{v}\cdot\operatorname{grad} g \\ g\mathbf{v}\cdot\operatorname{div}\mathbf{T} &= \operatorname{div}\{g\mathbf{v}\cdot\mathbf{T}\} - g\mathbf{T}\cdot\mathbf{D} - \mathbf{v}\cdot\mathbf{T}\operatorname{grad} g. \end{aligned}$$

Integrate now over the set $\Omega_\mu \times (0, t)$, with $t < \tau$ and

$$\Omega_\mu = \{\mathbf{x} \in \Omega: |\mathbf{x} - \mathbf{y}| \geqslant \mu > 0, \forall \mathbf{y} \in \partial\Omega\},$$

and choose μ in such a way that $\Omega_\mu \neq \varnothing$.

A simple application of the divergence theorem leads at once to

$$\begin{aligned}\frac{1}{2}\int_{\Omega_\mu} g\rho v^2\, d\Omega = {} & \frac{1}{2}\int_{\Omega_\mu} g_0\rho_0 v_0^2 d\Omega - \frac{1}{2}\int_0^t ds \int_{\Omega_\mu} g'_\delta \rho v^2 d\Omega - \int_0^t ds \int_{\Omega_\mu} g\mathbf{T}\cdot\mathbf{D}\, d\Omega \\ & - \frac{1}{2c}\int_0^t ds \int_{\Omega_\mu} g'_\delta q'(r)\, v^2 \mathbf{v}\cdot\mathbf{e}_r d\Omega \\ & + \frac{1}{c}\int_0^t ds \int_{\Omega_\mu} g'_\delta q'(r)\, \mathbf{v}\cdot\mathbf{T}\mathbf{e}_r d\Omega \\ & - \int_0^t ds \int_{\partial\Omega_\mu} g\left(\frac{1}{2}\rho v^2 \mathbf{v} - \mathbf{v}\cdot\mathbf{T}\right)\cdot\mathbf{n}_\mu d\Omega \end{aligned} \tag{5}$$

where $\mathbf{n}_\mu$ is the outward unit normal to $\partial\Omega_\mu$.

By virtue of the inequalities

$$-2\frac{q'(r)}{c}\mathbf{v}\cdot\mathbf{T}\mathbf{e}_r \leqslant \frac{1}{c}\rho v^2 + \frac{1}{c}[q'(r)]^2 T^2/\rho$$

$$2\mathbf{b}\cdot\mathbf{v} \leqslant \rho v^2 + b^2/\rho$$

and since, by $(4)_1$, $\exists M > 0$: $q'(r)\,|\mathbf{v}\cdot\mathbf{e}_r| \leqslant M$ on $\Omega \times [0, \tau]$, (5) yields

$$\begin{aligned}\frac{1}{2}\int_{\Omega_\mu} g\rho v^2 d\Omega \leqslant {} & \frac{1}{2}\int_{\Omega_\mu} g_0\rho_0 v_0^2\, d\Omega + \frac{1}{2}\left(\frac{M+1}{c} - 1\right)\int_0^t ds \int_{\Omega} g'_\delta \rho v^2 d\Omega \\ & + \frac{1}{2}\int_0^t ds \int_{\Omega_\mu} g\rho v^2 d\Omega + \int_0^t ds \int_{\Omega_\mu} g(b^2/2\rho - \mathbf{T}\cdot\mathbf{D})\, d\Omega \\ & + \frac{1}{2c}\int_0^t ds \int_{\Omega_\mu} g'_\delta [q'(r)]^2 T^2/\rho\, d\Omega \\ & - \int_0^t ds \int_{\partial\Omega_\mu} g\left\{\frac{1}{2}\rho v^2\mathbf{v} - \mathbf{v}\cdot\mathbf{T}\right\}\cdot\mathbf{n}_\mu d\Sigma. \end{aligned} \tag{6}$$

Choose now $c > M + 1$. An obvious application of Grönwall's lemma and passage to the limit $\mu \to 0$ give

$$\begin{aligned}\frac{1}{2}\int_{\Omega} g\rho v^2 d\Omega \leqslant e^t \Bigg\{ & \int_{\Omega} g_0\rho_0 v_0^2\, d\Omega + \int_0^t e^{-s} ds \Bigg[\int_{\Omega}\Bigg(b^2/2\rho - \mathbf{T}\cdot\mathbf{D} \\ & + \frac{1}{2c} g'_\delta [q'(r)]^2 T^2/\rho\Bigg) d\Omega - \int_{\partial\Omega} g\left(\frac{1}{2}\rho v^2\mathbf{v} - \mathbf{v}\cdot\mathbf{T}\right)\cdot\mathbf{n}\, d\Sigma\Bigg]\Bigg\}. \end{aligned} \tag{7}$$

Observe now that, as $\delta \to 0$, g tends to the characteristic function of the set χ_0, and the passage to the limit $\delta \to 0$ is permissible in (7). Moreover

$$\frac{1}{c}\int_0^t ds \int_\Omega g'_\delta [q'(r)]^2 T^2/\rho \, d\Omega \xrightarrow{\delta \to 0} \int_0^t ds \int_{\Omega \cap \partial S_{R_s}} q'(r)\, T^2/\rho \, d\Sigma$$

where we have put $R_s = q^{-1}[q(R) + c(\tau - s)]$. Thus, by letting $\mu \to 0$, (7) leads to

$$\begin{aligned}\tfrac{1}{2}\int_{\Omega \cap S_{R_t}} \rho v^2 d\Omega \leqslant e^t \Bigg\{ \tfrac{1}{2}\int_{\Omega \cap S_{R_0}} \rho_0 v_0^2 d\Omega + \int_0^t e^{-s} ds \Bigg[\int_{\Omega \cap S_{R_s}} (b^2/2\rho - \mathbf{T} \cdot \mathbf{D}) d\Omega \\ - \int_{\partial\Omega \cap S_{R_s}} (\tfrac{1}{2}\rho v^2 \mathbf{v} - \mathbf{v} \cdot \mathbf{T}) \cdot \mathbf{n} \, d\Sigma \Bigg] + \int_0^t ds \int_{\Omega \cap \partial S_{R_s}} o(R^{1-n}) d\Sigma \Bigg\}.\end{aligned}$$

This last relation assures us that ρv^2 and $\mathbf{T} \cdot \mathbf{D}$ are respectively summable on Ω and $\Omega \times (0, t)$, $\forall t \geqslant 0$. Indeed, letting $R \to +\infty$, we have

$$\begin{aligned}\tfrac{1}{2}\int_\Omega \rho v^2 d\Omega + \int_0^t ds \int_\Omega \mathbf{T} \cdot \mathbf{D} \, d\Omega \leqslant e^t \Bigg\{ \tfrac{1}{2}\int_\Omega \rho_0 v_0^2 d\Omega + \int_0^t e^{-s} \Bigg[\int_\Omega b^2/2\rho \, d\Omega \\ - \int_{\partial\Omega} (\tfrac{1}{2}\rho v^2 \mathbf{v} - \mathbf{v} \cdot \mathbf{T}) \cdot \mathbf{n} \, d\Sigma \Bigg] \Bigg\}. \qquad (8)\end{aligned}$$

Now, let f be any smooth function of r. Multiplying both sides of $(3)_1$ by $f\mathbf{v}$ and operating in the same way as in the derivation of (5), we get, $\forall R > 0$,

$$\begin{aligned}\tfrac{1}{2}\int_{\Omega_\mu \cap S_R} f \rho v^2 d\Omega = \Bigg\{ \tfrac{1}{2}\int_{\Omega_\mu \cap S_R} f \rho_0 v_0^2 d\Omega + \int_0^t ds \Bigg[\int_{\Omega_\mu \cap S_R} (\tfrac{1}{2}\rho v^2 \mathbf{v} \operatorname{grad} f \\ - f\mathbf{T} \cdot \mathbf{D} - \mathbf{v} \cdot \mathbf{T} \operatorname{grad} f + \mathbf{b} \cdot \mathbf{v}) \, d\Omega \\ - \int_{\partial\Omega_\mu \cap S_R} f(\tfrac{1}{2}\rho v^2 \mathbf{v} - \mathbf{v} \cdot \mathbf{T}) \cdot \mathbf{n} \, d\Sigma \\ - \int_{\Omega_\mu \cap \partial S_R} f(\tfrac{1}{2}\rho v^2 \mathbf{v} - \mathbf{v} \cdot \mathbf{T}) \cdot e_r \, d\Sigma \Bigg] \Bigg\}. \qquad (9)\end{aligned}$$

By choosing $f \equiv 1$ in (9) and by virtue of the hypotheses and (8), we easily argue that the last integral, as $R \to +\infty$, is certainly finite. Then, let us choose $f = [q(r)]^{-\alpha}$, $\alpha > 0$. Since

$$-\int_{\Omega_\mu \cap S_R} \rho v^2 \mathbf{v} \operatorname{grad} f d\Omega$$

$$= \alpha \int_{\Omega_\mu \cap S_R} [q(r)]^{-\alpha-1} q'(r) \rho v^2 \mathbf{v} \cdot \mathbf{e}_r \, d\Omega$$

$$\leqslant K\alpha \int_{\Omega_\mu \cap S_R} \rho v^2 \, d\Omega$$

$$-\int_{\Omega_\mu \cap S_R} \mathbf{v} \cdot \mathbf{T} \operatorname{grad} f d\Omega$$

$$\leqslant \tfrac{1}{2}\alpha \int_{\Omega_\mu \cap S_R} \rho v^2 \, d\Omega$$

$$+\frac{1}{2}\alpha \int d\gamma \int_0^R [q(r)]^{-2(\alpha+1)} [q'(r)]^2 T^2 / \rho r^{n-1} \, dr$$

$$\leqslant \frac{1}{2}\alpha \int_{\Omega_\mu \cap S_R} \rho v^2 d\Omega + \frac{K\alpha}{2(2\alpha+1)} \{ [q(0)]^{-2\alpha-1} - [q(r)]^{-2\alpha-1} \}$$

where K is a suitable positive constant and (r, γ) is the spherical coordinate system associated to $\{\mathbf{0}, \mathbf{e}_i\}$, the theorem is finally achieved, by letting in (9) first $\alpha \to 0$, then $R \to +\infty$ and $\mu \to 0$.

3. Concluding Remarks

Our theorem may be easily proved for any unbounded material part P of B [1].

If we specify the material, we may obtain sharper results. A simple example is provided by an elastic flow, i.e., a fluid whose constitutive equation is $\mathbf{T} = -\pi(\rho)\,\mathbf{I}$. In this case assumption $(4)_2$ is no longer required. Moreover a very strong domain of influence theorem holds [3].

References

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